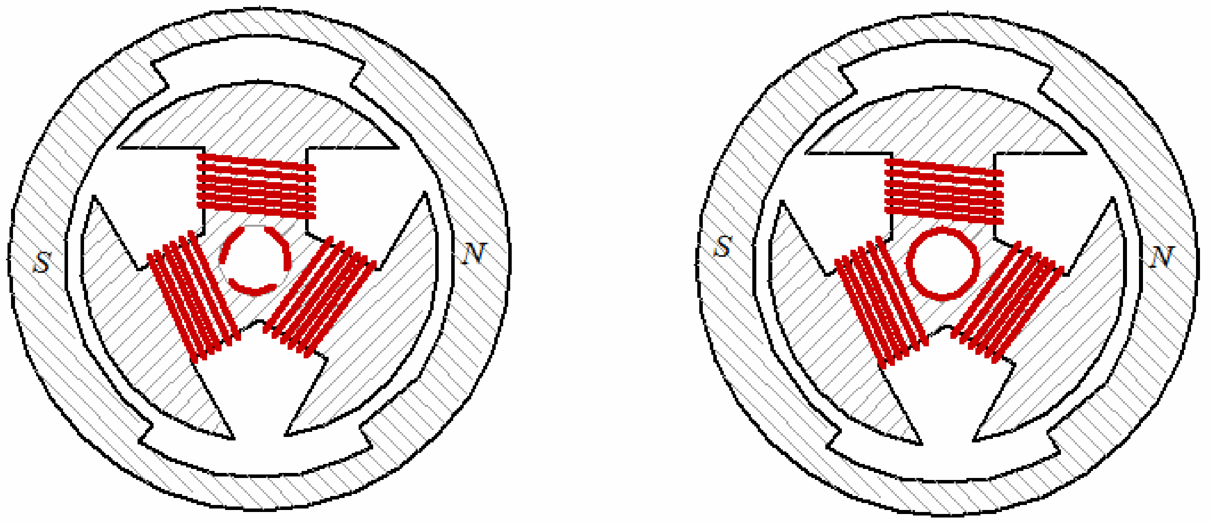
Phase advancing for DC electric motors

Although this is not an ambitious text, I wanted to investigate a DC motor in a more detailed way than it can be found in a typical internet-sourced overview article. After I learned that an idealized AC synchronous motor does not have a theoretical upper speed limit, I was motivated to investigate if the same can be true for a DC motor.

A DC motor and an AC synchronous motor are, I could argue, quite similar. The left picture below shows a simplified brushed DC motor, while the right picture shows a simplified synchronous AC motor. Can you tell the difference?



The difference is that the DC motor has commutator (the motor above has 3 commutator plates), while the AC motor has slip rings (the motor above has 3 slip rings). You must feed alternate voltage to an AC motor, while a DC motor uses its commutator to ‘alternatize’ the supplied DC voltage. In both cases, however, the motor winding ‘feels’ the AC voltage and carries AC current.

In practice you will not see AC motors built the way as depicted above. The AC motor will be made ‘inverted’ way, having windings on stator and the permanent magnet on rotor – this makes slip rings unneeded. The inversion does not, however, change the physics involved… Brushless DC motors (BLDC) are also made in this inverted way. If one examines a BLDC motor it can be seen that the mechanical part of the BLDC motor is pretty much identical to an AC synchronous motor.

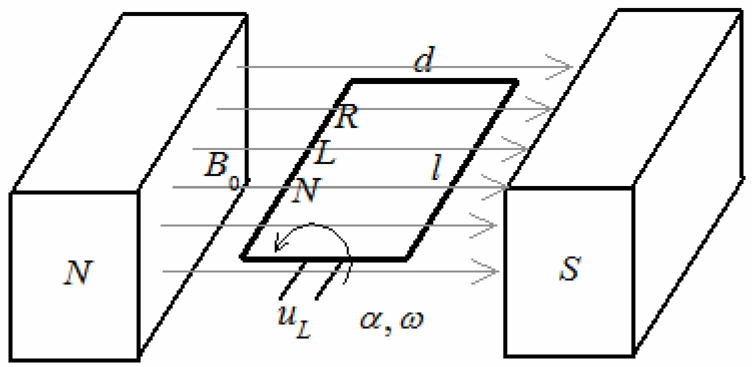
What is the real difference? The difference is in philosophy on how the voltage is supplied to motor winding. In the case of a DC motor, the phase of the voltage supplied to motor winding depends on the rotor angular position. In the case of an AC motor, the rotor angular position depends on the phase of the supplied voltage.

In the case of a brushed DC motor, it is the commutator that is responsible to convert the voltage supplied by ‘DC bus’ into the AC voltage and feed it to motor windings. In the case of brushless DC motor, instead of the commutator there is an inverter circuit that converts a voltage from DC bus into AC and feeds it to the windings. In both cases the phase of the voltage that is feed to the motor winding follows (is function of) the rotor angular position.

That said, it is obvious that you cannot have that much control over a DC motor speed and angular position as you can have it in the AC synchronous motor case. On the other hand, to control AC synchronous motor speed you have to be able to generate AC voltage of adjustable frequency, and this is not easy. To change speed of a DC motor you just need to change the supplied average DC voltage and this is considerably easier.

A simple DC motor model

I am going to make a DC motor model simple enough that even I can use math against it. My motor therefore consists of rectangular wire winding rotating inside a homogenous magnetic field in vacuum (this reminds me on the ‘spherical cows’ joke).

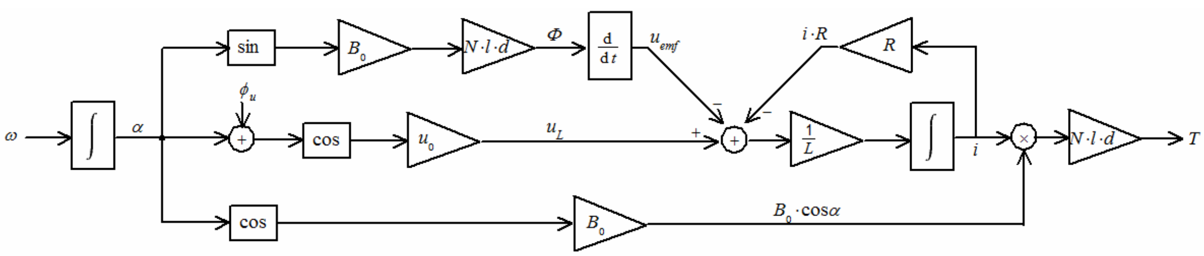


The wire winding has *N* turns, length *l*, width *d*, resistance *R* and inductance *L*. The magnetic field flux density is B_0. The AC voltage u_L is supplied to the loop – here it is not important how the AC voltage is obtained (by a commutator or by some electronics). It is only important that the supplied voltage u_L phase directly depends on the angular position of the loop (rotor). The angular position is α and the rotation speed is ω (rad/s).

Very dissimilar to a real motor, our motor winding voltage u_L is going to be a pure sine voltage of the amplitude u_0! The reason is because it makes the math easier for me. As said, the u_L phase follows the winding (rotor) angular position, but there can be, in a general case, some phase difference ϕ_u between the two. Therefore:

$$u_L = u_0 \cos(\alpha + \phi_u)$$

Further, I am not going to allow my rotor to rotate freely. Instead I will rotate it externally with some rotation speed ω. I am interested to see what torque does motor produce at certain rotation speed ω and input voltage amplitude u_0 and phase ϕ_u. Because I will rotate the rotor at steady speed ω, the angular position of the rotor is as simple as: $\alpha = \omega t$.



The above schematic shows our simple model. At the input we have rotor rotation speed ω that is steadily integrated to produce rotor position α. The rotor position α is then used in three ways. The middle branch produces the voltage u_L that is feed to the winding as described by previously mentioned formula.

The upper branch produces the back-electromotive force (BEMF) that is designated as u_{emf}. The u_{emf} is calculated as derivation of the flux through the winding multiplied by number of wire turns N, while flux is calculated as winding surface area ($l\ d$) multiplied by flux density B_0 and factor sin(α) to account for loop angle.

$$u_{emf} = \frac{\mathrm{d}}{\mathrm{d}t}(B_0\, l\, d\, N \sin\alpha) \qquad \Rightarrow \text{substitution } k = l\, d\, N \qquad \Rightarrow u_{emf} = \frac{\mathrm{d}}{\mathrm{d}t}(B_0\, k \sin\alpha)$$

The u_L and the u_{emf} are combined to drive current through the winding. Of course, the current also depends on winding inductance L and resistance R. We get a simple differential equation.

$$i = \frac{1}{L}\int (u_L - u_{emf} - i\,R)\,\mathrm{d}t$$

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = u_L - u_{emf} - i\,R$$

Finally, the output torque T is calculated as a multiplication of the winding current and the flux density (factored by $\cos\alpha$ to include only the component of the force that is tangential to rotor) and multiplied by winding physical dimensions and number of wire turns.

$$T = i\,B_0\,k\cos\alpha$$

As a basis for our further analysis, we first want to compute how the winding current i depends on the rotational speed ω, input voltage amplitude u_0 and input voltage phase ϕ. I am a lousy mathematician so I will help myself by using Laplace transform.

$$\left.\begin{array}{ll} \alpha = \omega t & \\ u_L = u_0\cos(\alpha + \phi_u) & \Rightarrow u_L = u_0\cos(\omega t + \phi_u) \\ u_{emf} = \frac{\mathrm{d}}{\mathrm{d}t}(B_0\, k \sin\alpha) & \Rightarrow u_{emf} = B_0\, k\,\omega\cos\omega t \\ L\frac{\mathrm{d}i}{\mathrm{d}t} = u_L - u_{emf} - iR & \end{array}\right\} \; L\frac{\mathrm{d}i}{\mathrm{d}t} = u_0\cos(\omega t + \phi_u) - B_0\, k\,\omega\cos\omega t - iR$$

$$\Downarrow \text{ (into laplace domain)}$$

$$L\,i\,s = u_0\frac{s\cos\phi_u - \omega\sin\phi_u}{s^2+\omega^2} - B_0\, k\,\omega\frac{s}{s^2+\omega^2} - iR$$

I was a bit sloppy here because we are not interested in transient part of the solution, but only in the steady state part of the solution. After some math gymnastics we have the steady state time-domain solution as:

$$i = \frac{u_0\, L\,\omega\sin\phi_u + u_0\, R\cos\phi_u - \omega\, R\, B_0\, k}{R^2 + L^2\omega^2}\cos\omega t + \frac{u_0\, L\,\omega\cos\phi_u - u_0\, R\sin\phi_u - L\,\omega^2 B_0\, k}{R^2 + L^2\omega^2}\sin\omega t$$

The obtained current is a sum of a cosine and sine part, and we can therefore conclude that it has a sine-wave form that, if needed, can be written as:

$$i = i_0 \cos(\omega t - \phi_i) \qquad \text{where} \qquad i_0 = \sqrt{\frac{u_0^2 - 2\, u_0\, \omega \cos\phi_u\, B_0\, l\, d + \omega^2 B_0^2 k^2}{R^2 + L^2 \omega^2}}$$

$$\phi_i = \arctan \frac{u_0\, L\, \omega \cos\phi_u - u_0\, R \sin\phi_u - L\, \omega^2 B_0\, k}{u_0\, L\, \omega \sin\phi_u + u_0\, R \cos\phi_u - \omega\, R\, B_0\, k}$$

This is a very complex solution, but some things can still be concluded. For example, when ω goes toward extremes the current becomes:

for very small ω

$$i = \frac{u_0}{R} \cos(\omega t + \phi_u)$$

for very large ω

$$i = \frac{B_0\, k}{L} \cos\left(\omega t + \frac{\pi}{2}\right) \qquad \text{if } u_0 > \frac{R\, B_0\, k}{L \sin\phi_u}$$

$$i = \frac{B_0\, k}{L} \cos\left(\omega t - \frac{\pi}{2}\right) \qquad \text{if } u_0 < \frac{R\, B_0\, k}{L \sin\phi_u}$$

At very high frequencies the current amplitude approaches some constant value. There is something fishy happening to current phase – depending on the input voltage it can move to $+\pi/2$ or to $-\pi/2$.

In the further analysis I will be mostly interested in two things:

1. What is the upper motor speed depending on the input voltage phase ϕ_u? More generally, what is the ϕ_u that gives the largest motor torque for any given voltage amplitude u_0 and rotation speed ω?
2. What is the voltage phase ϕ_u that maximizes motor efficiency (actually, the ϕ_u that develops smallest winding current for some given torque T)?

Calculating the motor torque

We can now substitute the obtained current into the torque equation and we get:

$$T = B_0 k \left[\frac{u_0\, L\, \omega \sin\phi_u + u_0\, R \cos\phi_u - \omega\, R\, B_0\, k}{R^2 + L^2 \omega^2} \cos^2 \omega t + \frac{1}{2} \frac{u_0\, L\, \omega \cos\phi_u - u_0\, R \sin\phi_u - L\, \omega^2 B_0\, l\, d}{R^2 + L^2 \omega^2} \sin 2\omega t \right]$$

The above equation shows the torque, as a function of time, for our DC motor model. We are not so much interested about how torque varies in time (although even this may be interesting if we consider how smooth does motor run), but we are more interested about the average torque motor gives at certain speed ω. To find this out, we will integrate the above torque formula during one rotation period and divide the value by the rotation period.

$$T_{avg} = \frac{B_0\, k}{2} \cdot \frac{u_0\, L\, \omega \sin\phi_u + u_0\, R \cos\phi_u - \omega\, R\, B_0\, k}{R^2 + L^2 \omega^2}$$

Now we have something useful – a formula that tells how the developed torque depends on the rotation speed ω, but also on the input voltage amplitude u_0 and phase ϕ_u. Let's play a bit with the formula…. For very low and for very high rotation speeds it degenerates into:

$$T_{avg} = \frac{B_0 k}{2} \frac{u_0 \cos \phi_u}{R} \qquad \text{for very small } \omega$$

$$T_{avg} = 0 \qquad \text{for verey large } \omega$$

We see, unsurprisingly, that at very low speeds (starting torque) the torque will be at maximum if the supplied voltage is exactly in phase with the rotor angular position (ϕ_u=0).

We want to find the rotation speed at which the torque falls to zero. In other words, we want to find the rotation speed at which the rotor would not accelerate any more – the unloaded motor top speed. Obviously, the torque falls to zero when the following is true:

$$u_0 L \omega \sin \phi_u + u_0 R \cos \phi_u - \omega R B_0 k = 0$$

$$\Rightarrow \quad \omega_{top} = \frac{u_0 R \cos \phi_u}{R B_0 k - u_0 L \sin \phi_u}$$

The obtained formula for unloaded motor top speed is very interesting because we can conclude that there are cases when the unloaded motor does not have limited top speed. This happens when.

$$R B_0 k \leq u_0 L \sin \phi_u$$

The above condition can only be true if ϕ_u is between 0 and π. It can never be true if:

$$u_0 < \frac{R B_0 k}{L} = u_{unlimited\_speed}$$

Therefore, there is some minimal input voltage amplitude that is required to achieve unlimited top speed of an unloaded motor. If the voltage is above this limit, then by properly adjusting ϕ_u, you 'can' achieve the unlimited speed. If the voltage is below this limit, then the unlimited speed cannot be achieved… This is the result I expected so I can conclude that there is no difference between DC and AC synchronous motors regarding this fact.

In real world there is no such thing as 'unloaded motor'. Every motor has some load, like bearing friction, air friction, iron looses. As a result, the upper speed limit always exists… very fortunate, I would say.

We are also going to find the ϕ_u that maximizes the motor torque. To find such extreme, we must find the derivative of the torque formula by ϕ_u.

$$\frac{\mathrm{d}\,T_{avg}}{\mathrm{d}\,\phi} = \frac{B_0 k u_0}{2(R^2 + L^2\omega^2)}(L\omega\cos\phi_u - R\sin\phi_u) = 0$$

$$\Rightarrow \qquad L\omega\cos\phi_u = R\sin\phi_u$$

$$\Rightarrow \qquad \boxed{\mathrm{tg}\,\phi_u = \frac{L\omega}{R}}$$

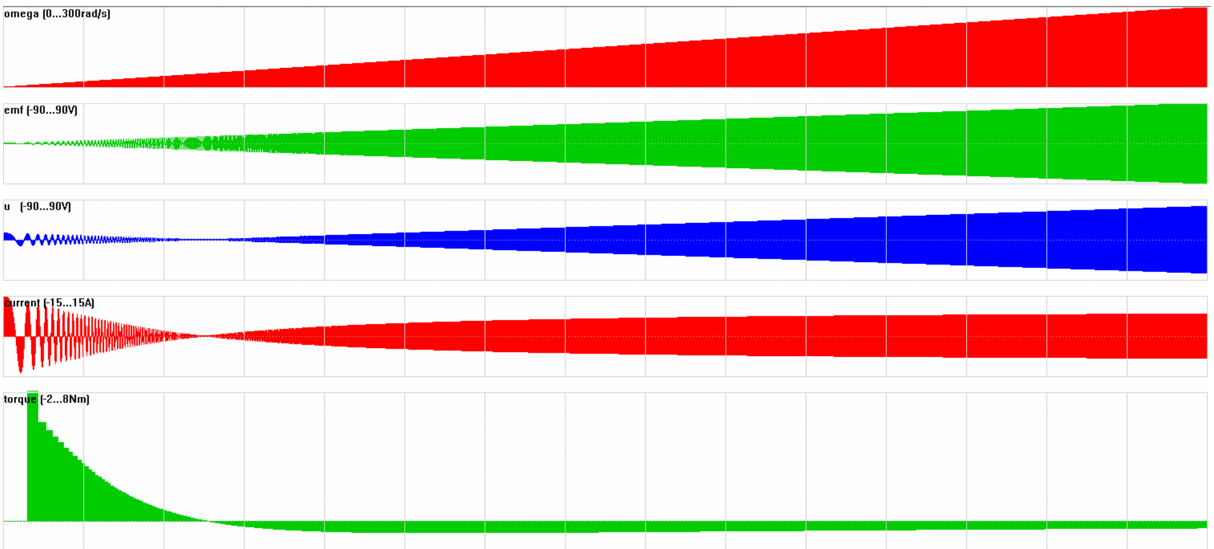
$$\phi_{u\ max\_torque} = 0;\ \text{ for small } \omega$$

$$\phi_{u\ max\_torque} = \frac{\pi}{2};\ \text{ for high } \omega$$

What we obtained is the formula for input-voltage phase advance ϕ_u that will give the motor maximal torque for any input voltage amplitude. At low motor speed the optimal ϕ_u is near zero, at high motor speeds the optimal ϕ_u goes toward π/2. This in practice means that if we want to extract maximum torque (and thus also the power) at high speeds, we need to advance the phase of the feed voltage in respect to rotor angular position.

I made a computer simulation of the described motor model. Below we can see results for a motor that has a 30-turn loop winding, dimensions 10x10cm. The winding resistance is 1ohm and winding inductance is 30mH. The stator flux density is 1T. The motor is supplied by 15V (as explained earlier, our special 'commutator' provides sine wave of the 15V amplitude to the motor winding).

The first set of graphs, shown below, displays the case when the input voltage phase ϕ_u was always zero (no phase advance as the speed increases).



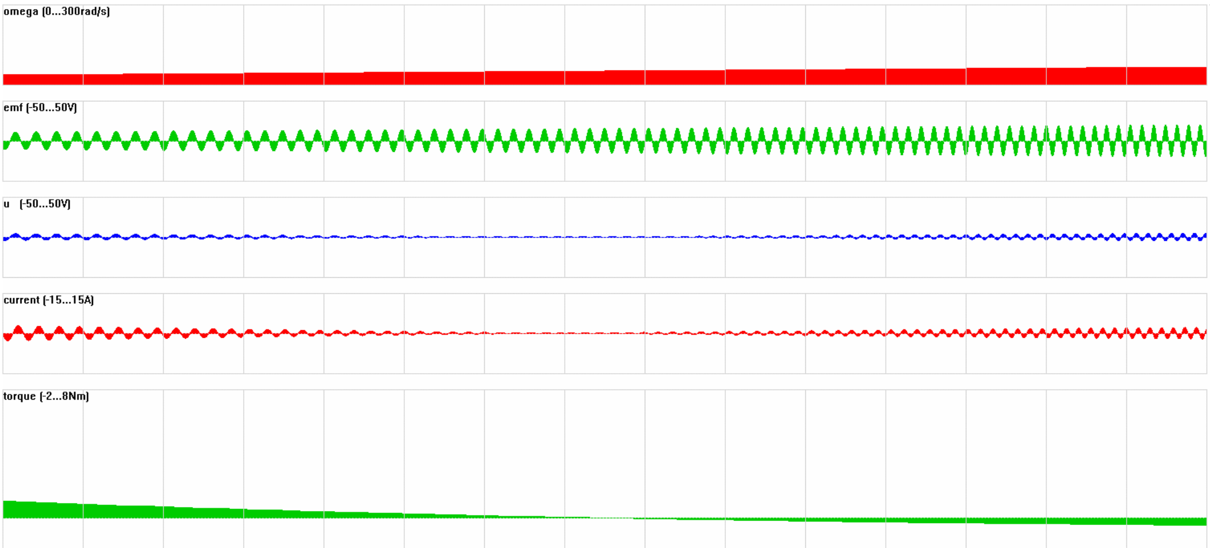
The topmost graph shows that the motor rotation speed ω (the input into the simulation) was very slowly increased from zero to about 300 rad/s (2866rpm). Recall that we are forcing this rotation speed by some external means and we are measuring the torque provided by motor at any given rotation speed.

The second graph shows the motor back-emf. As expected, the back-emf linearly increases as the motor rotation speed increases. The third graph shows some voltage 'u' that is combination of the input voltage u_L and the back-emf: $u=u_L-u_{emf}$. This is the voltage that drives current through the winding. As you can see, at one moment the back-emf becomes equal to the input voltage and no current is driven through the winding.

The fourth graph shows the motor winding current. The bottom graph shows the motor torque. As it can be seen, at one moment the motor torque drops to zero and then enters into negative region, meaning that the motor is working as a generator. [The torque graph looks a bit quantized because it shows the average torque value that is only calculated once per motor revolution.]

The interesting thing to note on above graphs is that current rises to a limited value and then does not rise any more despite the fact that voltage 'u' is still rising. It is because inductance of the winding provides impedance to high-frequency currents. Another interesting thing is that at very high speeds the motor torque slowly moves toward zero (from below).

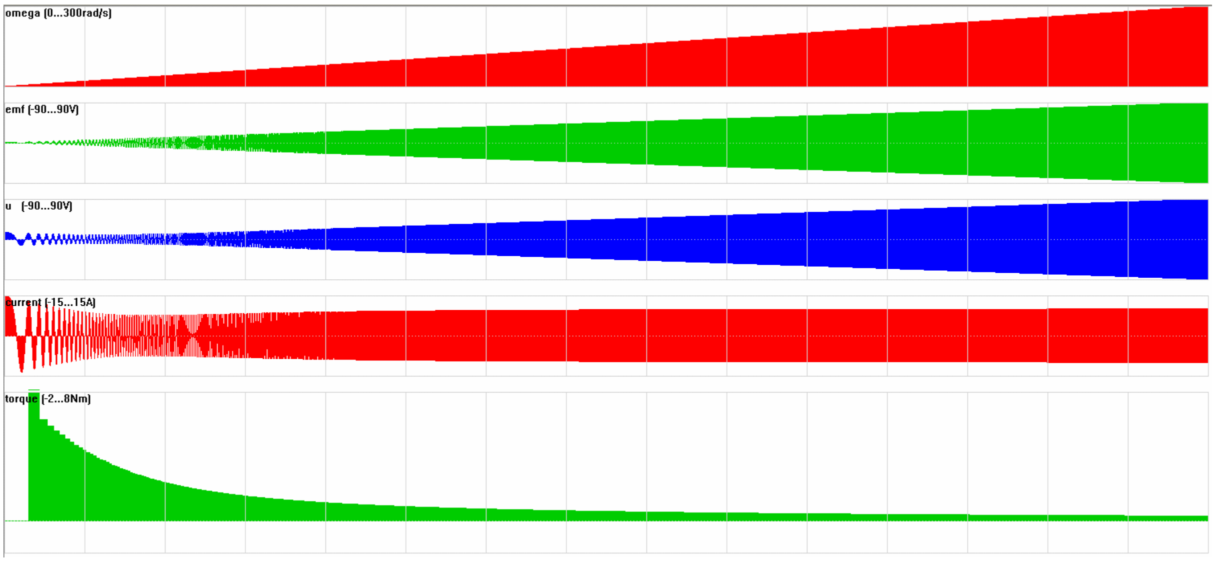
The second set of graphs shows the same situation, but zoomed-in in the region where the torque crosses zero line.



The interesting thing to note is how, before the event, the 'u' and current are in the same phase as the back-emf, while after the event they are in anti-phase. Another observable thing is that even at these relatively low speeds the current phase lags a bit after the 'u' due to winding inductance.

The third sets of graph shows the same situation as in the first case, but now the input voltage phase is advanced by the 'max-torque' formula as we calculated earlier:

$$\phi_u = \arctan \frac{L\,\omega}{R}$$



The first two graphs are without change, as expected. The $u=u_L-u_{emf}$ now never reaches zero because there is now some phase difference between u_L and u_{emf}. As a result, neither the current reaches zero. The torque curve is always positive (although small at high speeds) – we used 15V input voltage amplitude which is higher than the limit voltage needed for ‘infinite speed’ (10V).

Computing motor efficiency

The motor torque (and thus also its speed, depending on motor load) can be regulated by both, by changing the input voltage or by changing (advancing/retarding) the phase of the voltage feed to the rotor in respect to rotor position. However, if you are also interested in efficiency then you will not be satisfied with just any phase&amplitude pair that gives the required torque. There must be the phase&amplitude solution that gives the best motor efficiency. That solution was not obvious to me.

It is easy to find that the highest efficiency (that is, the maximal torque with the lowest winding current amplitude) is obtained if the current is at maximum when rotor position α is zero. (In this position the force generated in winding wires is tangential to rotor this maximizes torque). This can be easily shown mathematically by integrating the torque for one rotation… Anyway, the required ‘efficient current’ should therefore only have it cos(ωt) component, while the sin(ωt) should be zero.

$$i = \frac{u_0 L\omega \sin\phi_u + u_0 R\cos\phi_u - \omega R B_0 k}{R^2 + L^2\omega^2}\cos\omega t + \underbrace{\frac{u_0 L\omega\cos\phi_u - u_0 R\sin\phi_u - L\omega^2 B_0 k}{R^2 + L^2\omega^2}}_{\substack{=0 \\ \text{max. efficiency condition}}}\sin\omega t$$

There exist a trivial solution, ϕ_u=0, $u_0=\omega B_0 k$, that does not interest us because it gives zero current and no torque. Further, no above condition can exist for high ω if u_0 is limited by any practical reason.

What I will do is that I will extract the u_0 from our torque formula and substitute it into the above condition. That is how I 'teach' our condition to respect some given torque that must be developed by the motor.

$$T_{avg} = \frac{B_0 k}{2}\cdot\frac{u_0 L\omega\sin\phi_u + u_0 R\cos\phi_u - \omega R B_0 k}{R^2+L^2\omega^2} \quad\Rightarrow\quad u_0 = \frac{2\frac{R^2+L^2\omega^2}{kB_0}T_{avg} + \omega R B_0 k}{L\omega\sin\phi_u + R\cos\phi_u}$$

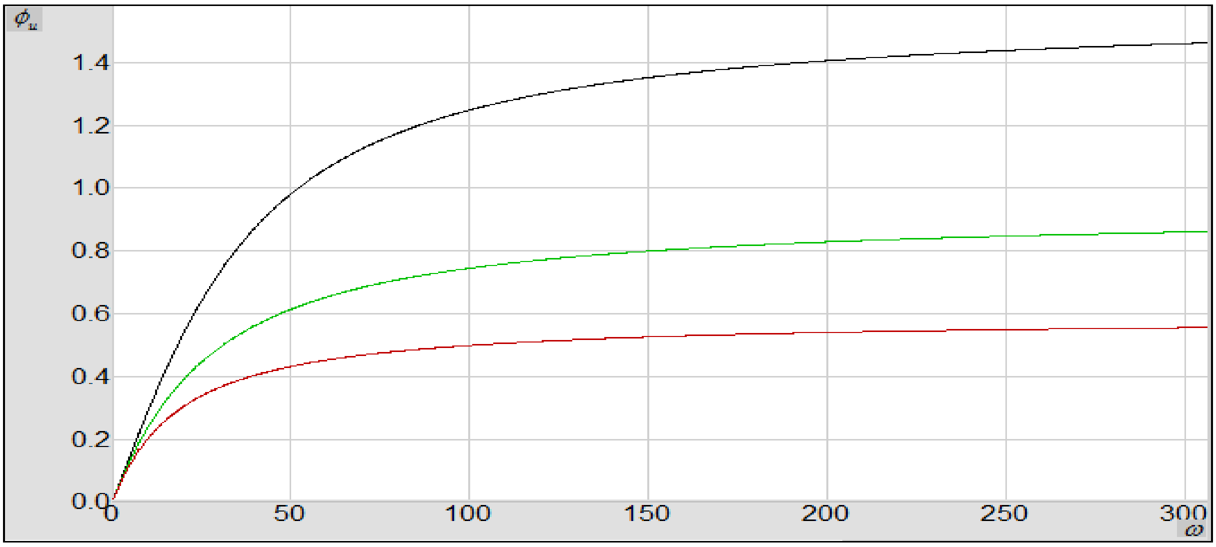
$$\frac{u_0 L\omega\cos\phi_u - u_0 R\sin\phi_u - L\omega^2 B_0 k}{R^2+L^2\omega^2} = 0 \quad\Rightarrow\quad u_0(L\omega\cos\phi_u - R\sin\phi_u) = L\omega^2 B_0 k$$

After quite some gymnastics I obtained the formula that tells what should be the input voltage phase to run motor efficiently when delivering certain torque at certain speed.

$$\operatorname{ctg}\phi_u = \frac{B_0^2 k^2}{2\,T_{avg} L} + \frac{R}{L\omega}$$

For very low rotation speeds, the optimal phase ϕ_u is zero. As the speed increases, the optimal phase increases toward some value smaller than $\pi/2$. Note that as the speed increases toward infinity, also the voltage amplitude u_0 must be increased toward infinity to keep producing the desired torque.

The input voltage phase advance ϕ_u needed for efficient drive is always smaller than input voltage phase advance ϕ_u needed to obtain maximum torque. The above examples shows the phase advance needed for maximum torque (black), the phase advance needed to efficiently develop 2Nm (green) and the phase advance needed to efficiently develop 1Nm (red).



If the load torque is not constant with the speed increase (a fan or a pump, for example) then the best-efficiency phase advance curve might have somewhat different shape.

While drawing our best-efficiency graphs our voltage amplitude was not limited, but in reality it will not be possible to obtain the best-efficiency working point at high rotation speeds because we will not be able to provide voltage above some level. From this point on, as the rotation speed continues to rise, you might still provide the desired torque by advancing the input voltage phase even further (departing from the best-efficiency curve toward the max-torque curve) until you reach the max-torque curve. Once you advanced the phase to a max-torque curve, and if the rotation speed continues to rise, it will not be possible to provide the desired torque any more.

Practical considerations

First, our model motor is far from a real-life motor. A real life motor is made of iron and iron is far from being a linear conductor of magnetic field. This might be the most important reason why our considerations here might only be of limited practical value. Second, we considered sine-wave voltage supply to the rotor winding, while in reality it would be more squarish or pulse-like.

If we are talking about brushed DC motors, a small motor will very likely have fixed brush position without possibility to regulate the input voltage phase. Even if not, its load will probably not have a predetermined load-curve (torque as a function of rotation speed) and you will needed to have some feedbacks, like speed and/or current measurements, to implement optimal phase-advancing.

The best candidate for the phase-advancement control, as it is described in this document, is a brushless DC motor. In this case we must have rotor position sensors anyway (or deduce the motor position from current measurements) and so we only need powerful-enough processor to adjust the phase advance.

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